

On the Efficiency of the Proportional Allocation Mechanism for Divisible Resources

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Abstract

We study the efficiency of the *proportional allocation mechanism*, that is widely used to allocate divisible resources. Each agent submits a bid for each divisible resource and receives a fraction proportional to her bids. We quantify the inefficiency of Nash equilibria by studying the Price of Anarchy (PoA) of the induced game under complete and incomplete information. When agents' valuations are concave, we show that the Bayesian Nash equilibria can be arbitrarily inefficient, in contrast to the well-known $4/3$ bound for pure equilibria [12]. Next, we upper bound the PoA over Bayesian equilibria by 2 when agents' valuations are subadditive, generalizing and strengthening previous bounds on lattice submodular valuations. Furthermore, we show that this bound is tight and cannot be improved by any *simple* or *scale-free* mechanism. Then we switch to settings with budget constraints, and we show an improved upper bound on the PoA over coarse-correlated equilibria. Finally, we prove that the PoA is *exactly* 2 for pure equilibria in the polyhedral environment.

1 Introduction

Allocating network resources, like bandwidth, among agents is a canonical problem in the network optimization literature. A traditional model for this problem was proposed by Kelly [14], where allocating these infinitely divisible resources is treated as a market with prices. More precisely, agents in the system submit bids on resources to express their willingness to pay. After soliciting the bids, the system manager prices each resource with an amount equal to the sum of bids on it. Then the agents buy portions of resources proportional to their bids by paying the corresponding prices. This mechanism is known as the *proportional allocation mechanism* or Kelly's mechanism in the literature.

The proportional allocation mechanism is widely used in network pricing and has been implemented for allocating computing resources in several distributed systems [5]. In practice, each agent has different interests for different subsets and fractions of the resources. This can be expressed via a *valuation* function of the resource allocation vector, that is typically private knowledge to each agent. Thus, agents may bid strategically to maximize their own utilities, i.e., the difference between their valuations and payments. Johari and Tsitsiklis [12] observed that this strategic bidding in the proportional allocation mechanism leads to inefficient allocations, that do not maximize social welfare. On the other hand, they showed that this efficiency loss is bounded when agents' valuations are concave. More specifically, they proved that the proportional allocation game admits a *unique pure* equilibrium with Price of Anarchy (PoA) [15] at most $4/3$.

An essential assumption used by Johari and Tsitsiklis [12] is that agents have complete information of each other's valuations. However, in many realistic scenarios, the agents are only partially

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informed. A standard way to model incomplete information is by using the Bayesian framework, where the agents' valuations are drawn independently from some publicly known distribution, that in a sense, represents the agents' beliefs. A natural question is whether the efficiency loss is still bounded in the Bayesian setting. We give a negative answer to this question by showing that the PoA over Bayesian equilibria is at least $\sqrt{m}/2$, where m is the number of resources. This result complements the current study by Caragiannis and Voudouris [2], where the PoA of single-resource proportional allocation games is shown to be at most 2 in the Bayesian setting.

Non-concave valuation functions were studied by Syrgkanis and Tardos [20] for both complete and incomplete information games. They showed that, when agents' valuations are lattice-submodular, the PoA for coarse correlated and Bayesian Nash equilibria is at most 3.73, by applying their general smoothness framework. In this paper, we study subadditive valuations [8] that is a superclass of lattice submodular functions. We prove that the PoA over Bayesian Nash equilibria is at most 2. Moreover, we show optimality of the proportional allocation mechanism, by showing that this bound is tight and cannot be improved by any *simple* mechanism, as defined in the recent framework of Roughgarden [19]¹, or any *scale-free* mechanism².

Next, we switch to the setting where agents are constrained by budgets, that represent the maximum payment they can afford. We prove that the PoA of the proportional allocation mechanism is at most $1 + \phi \approx 2.618$, where ϕ is the golden ratio. The previously best known bound was 2.78 and for a single resource due to [2]. Finally, we consider the polyhedral environment that was previously studied by Nguyen and Tardos in [16], where they proved that pure equilibria are at least 75% efficient with concave valuations. We prove that the PoA is exactly 2 for agents with subadditive valuations.

Related Work. The efficiency of the proportional allocation mechanism has been extensively studied in the literature of network resource allocation. Besides the work mentioned above, Johari and Tsitsiklis [13] studied a more general class of scale-free mechanisms and proved that the proportional allocation mechanism achieves the best PoA in this class. Zhang [21] and Feldman et al. [10] studied the efficiency and fairness of the proportional allocation mechanism, when agents aim at maximizing non quasi-linear utilities subject to budget constraints. Correa, Schulz and Stier-Moses [6] showed a relationship in the efficiency loss between proportional allocation mechanism and non-atomic selfish routing for not necessarily concave valuation functions.

There is a line of research studying the PoA of simple auctions for selling indivisible goods (see [1, 3, 11, 20]). Recently, Feldman et al. [9] showed tighter upper bounds for simultaneous first and second price auctions when the agents have subadditive valuations. Christodoulou et al. [4] showed matching lower bounds for simultaneous first price auctions, and Roughgarden [19] proved general lower bounds for the PoA of all simple auctions, by using the corresponding computational or communication lower bounds of the underlying allocation problem.

2 Preliminaries

There are n *agents* who compete for m *divisible resources* with *unit* supply. Every agent $i \in [n]$ has a valuation function $v_i : [0, 1]^m \rightarrow \mathbb{R}_+$, where $[n]$ denotes the set $\{1, 2, \dots, n\}$. The valuations are normalized as $v_i(\mathbf{0}) = 0$, and monotonically non-decreasing, that is, for every $\mathbf{x}, \mathbf{x}' \in [0, 1]^m$, where $\mathbf{x} = (x_j)_j$, $\mathbf{x}' = (x'_j)_j$ and $\forall j \in [m] \ x_j \leq x'_j$, we have $v_i(\mathbf{x}) \leq v_i(\mathbf{x}')$. Let $\mathbf{x} + \mathbf{y}$ be the componentwise sum of two vectors \mathbf{x} and \mathbf{y} .

¹In a simple mechanism, the agents' action space should be at most sub-doubly-exponential in m .

²The basic property of a scale-free mechanism is that, if every bid is scaled by the same constant, the outcome remains unchanged (we refer the reader to Section 4.3 for the complete definition).

Definition 1. A function $v : [0, 1]^m \rightarrow \mathbb{R}_{\geq 0}$ is subadditive if, for all $\mathbf{x}, \mathbf{y} \in [0, 1]^m$, such that $\mathbf{x} + \mathbf{y} \in [0, 1]^m$, it is $v(\mathbf{x} + \mathbf{y}) \leq v(\mathbf{x}) + v(\mathbf{y})$.

Remark 2. Lattice submodular functions used in [20] are subadditive (see Section 4). In the case of a single variable (single resource), any concave function is subadditive; more precisely, concave functions are equivalent to lattice submodular functions in this case. However, concave functions of many variables may not be subadditive [18].

In the *Bayesian* setting, the valuation of each agent i is drawn from a set of possible valuations V_i , according to some known probability distribution D_i . We assume that D_i 's are independent, but not necessarily identical over the agents.

A mechanism can be represented by a tuple (\mathbf{x}, \mathbf{q}) , where \mathbf{x} specifies the allocation of resources and \mathbf{q} specifies the agents' payments. In the mechanism, every agent i submits a non-negative bid b_{ij} for each resource j . The proportional allocation mechanism determines the allocation $x_i = (x_{ij})_j$ and payment q_i , for each agent i , as follows: $x_{ij} = \frac{b_{ij}}{\sum_{k \in [n]} b_{kj}}$, $q_i = \sum_{j \in [m]} b_{ij}$. When all agents bid 0, the allocation can be defined arbitrarily, but consistently.

Nash Equilibrium. We denote by $\mathbf{b} = (b_1, \dots, b_n)$ the strategy profile of all agents, where $b_i = (b_{i1}, \dots, b_{im})$ denotes the pure bids of agent i for the m resources. By $\mathbf{b}_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$ we denote the strategies of all agents except for i . Any *mixed, correlated, coarse correlated or Bayesian strategy* B_i of agent i is a probability distribution over b_i . For any strategy profile \mathbf{b} , $\mathbf{x}(\mathbf{b})$ denotes the allocation and $\mathbf{q}(\mathbf{b})$ the payments under the strategy profile \mathbf{b} . The *utility* u_i of agent i is defined as the difference between her valuation for the received allocation and her payment: $u_i(\mathbf{x}(\mathbf{b}), \mathbf{q}(\mathbf{b})) = u_i(\mathbf{b}) = v_i(x_i(\mathbf{b})) - q_i(\mathbf{b})$.

Definition 3. A bidding profile \mathbf{B} forms the following equilibrium if for every agent i and all bids b'_i :

Pure Nash equilibrium: $\mathbf{B} = \mathbf{b}$, $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$.

Mixed Nash equilibrium: $\mathbf{B} = \times_i B_i$, $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(b'_i, \mathbf{b}_{-i})]$.

Correlated equilibrium: $\mathbf{B} = (B_i)_i$, $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})|b_i] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(b'_i, \mathbf{b}_{-i})|b_i]$.

Coarse correlated equilibrium: $\mathbf{B} = (B_i)_i$, $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(b'_i, \mathbf{b}_{-i})]$.

Bayesian Nash equilibrium: $\mathbf{B}(\mathbf{v}) = \times_i B_i(v_i)$, $\mathbb{E}_{\mathbf{v}_{-i}, \mathbf{b}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{v}_{-i}, \mathbf{b}}[u_i(b'_i, \mathbf{b}_{-i})]$.

The first four classes of equilibria are in increasing order of inclusion. Moreover, any mixed Nash equilibrium is also a Bayesian Nash equilibrium.

Price of Anarchy (PoA). Our global objective is to maximize the sum of the agents' valuations for their received allocations, i.e., to maximize the *social welfare* $\text{SW}(\mathbf{x}) = \sum_{i \in [n]} v_i(x_i)$. Given the valuations, \mathbf{v} , of all agents, there exists an optimal allocation $\mathbf{o}^{\mathbf{v}} = \mathbf{o} = (o_1, \dots, o_n)$, such that $\text{SW}(\mathbf{o}) = \max_{\mathbf{x}} \text{SW}(\mathbf{x})$. By $o_i = (o_{i1}, \dots, o_{im})$ we denote the optimal allocation to agent i . For simplicity, we use $\text{SW}(\mathbf{b})$ and $v_i(\mathbf{b})$ instead of $\text{SW}(\mathbf{x}(\mathbf{b}))$ and $v_i(x_i(\mathbf{b}))$, whenever the allocation rule \mathbf{x} is clear from the context. We also use shorter notation for expectations, e.g. we use $\mathbb{E}_{\mathbf{v}}$ instead of $\mathbb{E}_{\mathbf{v} \sim \mathbf{D}}$, $\mathbb{E}[u_i(\mathbf{b})]$ instead of $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})]$ and $u(\mathbf{B})$ for $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u(\mathbf{b})]$ whenever \mathbf{D} and \mathbf{B} are clear from the context.

Definition 4. Let $\mathcal{I}([n], [m], \mathbf{v})$ be the set of all instances, i.e., $\mathcal{I}([n], [m], \mathbf{v})$ includes the instances for every set of agents and resources and any possible valuations that the agents might have for the resources. We define the pure, mixed, correlated, coarse correlated and Bayesian Price of Anarchy, PoA, as

$$\text{PoA} = \max_{I \in \mathcal{I}} \max_{\mathbf{B} \in \mathcal{E}(I)} \frac{\mathbb{E}_{\mathbf{v}}[\text{SW}(\mathbf{o})]}{\mathbb{E}_{\mathbf{v}, \mathbf{b} \sim \mathbf{B}}[\text{SW}(\mathbf{b})]},$$

where $\mathcal{E}(I)$ is the set of pure Nash, mixed Nash, correlated, coarse correlated or Bayesian Nash equilibria for the specific instance $I \in \mathcal{I}$, respectively³.

Budget Constraints. We also consider the setting where agents are budget-constrained. That is, the payment of each agent i cannot be higher than c_i , where c_i is a non-negative value denoting agent i 's budget. Following [2, 20], we use *Effective Welfare* as the benchmark: $\text{EW}(\mathbf{x}) = \sum_i \min\{v_i(x_i), c_i\}$. In addition, for any *randomized* allocation \mathbf{x} , the expected effective welfare is defined as: $\mathbb{E}_{\mathbf{x}}[\text{EW}(\mathbf{x})] = \sum_i \min\{\mathbb{E}_{\mathbf{x}}[v_i(x_i)], c_i\}$.

3 Concave Valuations

In this section, we show that for concave valuations on multiple resources, Bayesian equilibria can be arbitrarily inefficient. More precisely, we prove that the Bayesian PoA is $\Omega(\sqrt{m})$ in contrast to the constant bound for pure equilibria [12]. Therefore, there is a big gap between complete and incomplete information settings. We state our main theorem in this section as follows.

Theorem 5. When valuations are concave, the PoA of the proportional allocation mechanism for Bayesian equilibria is at least $\frac{\sqrt{m}}{2}$.

Proof. We consider an instance with m resources and 2 agents with the following concave valuations. $v_1(\mathbf{x}) = \min_j \{x_j\}$ and $v_2(\mathbf{x})$ is drawn from a distribution D_2 , such that some resource $j \in [m]$ is chosen uniformly at random and then $v_2(\mathbf{x}) = x_j/\sqrt{m}$. Let $\delta = 1/(\sqrt{m} + 1)^2$. We claim that $\mathbf{b}(\mathbf{v}) = (b_1, b_2(v_2))$ is a pure Bayesian Nash equilibrium, where $\forall j \in [m]$, $b_{1j} = \sqrt{\delta/m} - \delta$ and, if $j \in [m]$ is the resource chosen by D_2 , $b_{2j}(v_2) = \delta$ and for all $j' \neq j$ $b_{2j'} = 0$.

Under this bidding profile, agent 1 bids the same value for all resources, and agent 2 only bids positive value for a single resource associated with her valuation. Suppose that agent 2 has positive valuation for resource j , i.e., $v_2(\mathbf{x}) = x_j/\sqrt{m}$. Then the rest $m - 1$ resources are allocated to agent 1 and agents are competing for resource j . Bidder 2 has no reason to bid positively for any other resource. If she bids any value b'_{2j} for resource j , her utility would be $u_2(\mathbf{b}_1, b'_{2j}) = \frac{1}{\sqrt{m}} \frac{b'_{2j}}{b_{1j} + b'_{2j}} - b'_{2j}$, which is maximized for $b'_{2j} = \sqrt{\frac{b_{1j}}{\sqrt{m}}} - b_{1j}$. For $b_{1j} = \sqrt{\delta/m} - \delta$, the utility of agent 2 is maximized for $b'_{2j} = 1/(\sqrt{m} + 1)^2 = \delta$ by simple calculations.

Since $v_1(\mathbf{x})$ equals the minimum of \mathbf{x} 's components, agent 1's valuation is completely determined by the allocation of resource j . So the expected utility of agent 1 under \mathbf{b} is $\mathbb{E}_{v_2}[u_1(\mathbf{b})] = \frac{\sqrt{\delta/m} - \delta}{\sqrt{\delta/m} - \delta + \delta} - m(\sqrt{\delta/m} - \delta) = (1 - \sqrt{m\delta})^2 = \frac{1}{(\sqrt{m} + 1)^2} = \delta$. Suppose now that agent 1 deviates to $b'_1 = (b'_{11}, \dots, b'_{1m})$.

$$\begin{aligned} \mathbb{E}_{v_2}[u_1(b'_1, b_2)] &= \frac{1}{m} \sum_j \frac{b'_{1j}}{b'_{1j} + \delta} - \sum_j b'_{1j} = \frac{1}{m} \sum_j \left(\frac{b'_{1j}}{b'_{1j} + \delta} - m \cdot b'_{1j} \right) \\ &\leq \frac{1}{m} \sum_j \left(\frac{\sqrt{\delta/m} - \delta}{\sqrt{\delta/m}} - m \cdot (\sqrt{\delta/m} - \delta) \right) \\ &= \frac{1}{m} \sum_j \left(1 - 2\sqrt{m \cdot \delta} + m \cdot \delta \right) = \frac{1}{m} \sum_j \left(1 - \sqrt{m \cdot \delta} \right)^2 \\ &= \frac{1}{m} \sum_j \left(\frac{1}{\sqrt{m} + 1} \right)^2 = \delta = \mathbb{E}_{v_2}[u_1(\mathbf{b})]. \end{aligned}$$

³The expectation over \mathbf{v} is only needed for the definition of Bayesian PoA.

The inequality comes from the fact that $\frac{b'_{1j}}{b'_{1j} + \delta} - m \cdot b'_{1j}$ is maximized for $b'_{1j} = \sqrt{\delta/m} - \delta$. So we conclude that \mathbf{b} is a Bayesian equilibrium.

Finally we compute the PoA. The expected social welfare under \mathbf{b} is $\mathbb{E}_{v_2}[\text{SW}(\mathbf{b})] = \frac{\sqrt{\delta/m} - \delta}{\sqrt{\delta/m} - \delta + \delta} + \frac{1}{\sqrt{m}} \frac{\delta}{\sqrt{\delta/m} - \delta + \delta} = 1 - \sqrt{m\delta} + \sqrt{\delta} = \frac{2}{\sqrt{m+1}} < \frac{2}{\sqrt{m}}$. But the optimal social welfare is 1 by allocating to agent 1 all resources. So, $\text{PoA} \geq \frac{\sqrt{m}}{2}$. \square

4 Subadditive Valuations

In this section, we focus on agents with subadditive valuations. We prove that the proportional allocation mechanism is at least 50% efficient for coarse correlated equilibria and Bayesian Nash equilibria, i.e., $\text{PoA} \leq 2$. We further show that this bound is tight and cannot be improved by any simple or scale-free mechanism. Before proving our PoA bounds, we show that the class of subadditive functions is a superclass of lattice submodular functions.

Proposition 6. Any lattice submodular function v defined on $[0, 1]^m$ is subadditive.

Proof. It has been shown in [20] that for any lattice submodular function $v(x)$, $\frac{\partial^2 v(x)}{(\partial x_j)^2} \leq 0$ and $\frac{\partial^2 v(x)}{\partial x_j \partial x_{j'}} \leq 0$. So the function $\frac{\partial v}{\partial x_j}(x)$ is non-increasing monotone for each coordinate x_j . It suffices to prove that for any $\mathbf{x}, \mathbf{y} \in [0, 1]^m$, $v(\mathbf{x} + \mathbf{y}) - v(\mathbf{y}) \leq v(\mathbf{x}) - v(\mathbf{0})$. Let \mathbf{z}^k be the vector that $z_j^k = y_j$ if $j \leq k$ and $x_j + y_j$ otherwise. Note that $\mathbf{z}^0 = \mathbf{x} + \mathbf{y}$ and $\mathbf{z}^m = \mathbf{y}$. Similarly, we define \mathbf{w}^k to be the vector that $w_j^k = 0$ if $j \leq k$ and x_j otherwise. It is easy to see that $\mathbf{z}^k \geq \mathbf{w}^k$ for all $k \in [m]$. So we have,

$$\begin{aligned} v(\mathbf{x} + \mathbf{y}) - v(\mathbf{y}) &= \sum_{j \in [m]} v(\mathbf{z}^{j-1}) - v(\mathbf{z}^j) = \sum_{j \in [m]} \int_{y_j}^{x_j + y_j} \frac{\partial v}{\partial x_j}(t_j; \mathbf{z}_{-j}^j) dt_j \\ &\leq \sum_{j \in [m]} \int_{y_j}^{x_j + y_j} \frac{\partial v}{\partial x_j}(t_j - y_j; \mathbf{z}_{-j}^j) dt_j \leq \sum_{j \in [m]} \int_0^{x_j} \frac{\partial v}{\partial x_j}(s_j; \mathbf{w}_{-j}^j) ds_j = v(\mathbf{x}) - v(\mathbf{0}) \end{aligned}$$

The second equality is due to the definition of partial derivative and the inequalities is due to the monotonicity of $\frac{\partial v}{\partial x_j}(x)$. \square

4.1 Upper bound

A common approach to prove PoA upper bounds is to find a deviation with proper utility bounds and then use the definition of Nash equilibrium to bound agents' utilities at equilibrium. The bidding strategy described in the following lemma is for this purpose.

Lemma 7. Let \mathbf{v} be any subadditive valuation profile and \mathbf{B} be some randomized bidding profile. For any agent i , there exists a randomized bidding strategy $a_i(\mathbf{v}, \mathbf{B}_{-i})$ such that:

$$\sum_i u_i(a_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \frac{1}{2} \sum_i v_i(o_i^{\mathbf{v}}) - \sum_i \sum_j \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[b_{ij}].$$

Proof. Let p_{ij} be the sum of the bids of all agents except i on resource j , i.e., $p_{ij} = \sum_{k \neq i} b_{kj}$. Note that p_{ij} is a random variable that depends on $\mathbf{b}_{-i} \sim \mathbf{B}_{-i}$. Let P_i be the probability distribution of

$p_i = (p_{ij})_j$. Inspired by [9], we consider the bidding strategy $a_i(\mathbf{v}, \mathbf{B}_{-i}) = (o_{ij}^{\mathbf{v}} \cdot b'_{ij})_j$, where $b'_i \sim P_i$. Then, $u_i(a_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i})$ is

$$\begin{aligned}
& \mathbb{E}_{b'_i \sim P_i} \mathbb{E}_{p_i \sim P_i} \left[v_i \left(\left(\frac{o_{ij}^{\mathbf{v}} b'_{ij}}{o_{ij}^{\mathbf{v}} b'_{ij} + p_{ij}} \right)_j \right) - o_i^{\mathbf{v}} \cdot b'_i \right] \\
& \geq \frac{1}{2} \cdot \mathbb{E}_{p_i \sim P_i} \mathbb{E}_{b'_i \sim P_i} \left[v_i \left(\left(\frac{o_{ij}^{\mathbf{v}} b'_{ij}}{o_{ij}^{\mathbf{v}} b'_{ij} + p_{ij}} + \frac{o_{ij}^{\mathbf{v}} p_{ij}}{o_{ij}^{\mathbf{v}} p_{ij} + b'_{ij}} \right)_j \right) \right] - \mathbb{E}_{p_i \sim P_i} [o_i^{\mathbf{v}} \cdot p_i] \\
& \geq \frac{1}{2} \cdot \mathbb{E}_{p_i \sim P_i} \mathbb{E}_{b'_i \sim P_i} \left[v_i \left(\left(\frac{o_{ij}^{\mathbf{v}} (b'_{ij} + p_{ij})}{b'_{ij} + p_{ij}} \right)_j \right) \right] - \mathbb{E}_{p_i \sim P_i} [o_i^{\mathbf{v}} \cdot p_i] \\
& = \frac{1}{2} \cdot v_i(o_i^{\mathbf{v}}) - \sum_j \sum_{k \neq i} \mathbb{E}_{\mathbf{b} \sim \mathbf{B}} [o_{ij}^{\mathbf{v}} \cdot b_{kj}]
\end{aligned}$$

The first inequality follows by swapping p_{ij} and b'_{ij} and using the subadditivity of v_i . The second inequality comes from the fact that $o_{ij}^{\mathbf{v}} \leq 1$. The lemma follows by summing up over all agents and the fact that $\sum_{i \in [n]} o_{ij}^{\mathbf{v}} = 1$. \square

Theorem 8. The coarse correlated PoA of the proportional allocation mechanism with subadditive agents is at most 2.

Proof. Let \mathbf{B} be any coarse correlated equilibrium (note that \mathbf{v} is fixed). By Lemma 7 and the definition of the coarse correlated equilibrium, we have

$$\sum_i u_i(\mathbf{B}) \geq \sum_i u_i(a_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \frac{1}{2} \sum_i v_i(o_i) - \sum_i \sum_j \mathbb{E}[b_{ij}]$$

By rearranging terms, $\text{SW}(\mathbf{B}) = \sum_i u_i(\mathbf{B}) + \sum_i \sum_j \mathbb{E}[b_{ij}] \geq \frac{1}{2} \cdot \text{SW}(\mathbf{o})$. \square

Theorem 9. The Bayesian PoA of the proportional allocation mechanism with subadditive agents is at most 2.

Proof. Let \mathbf{B} be any Bayesian Nash Equilibrium and let $v_i \sim D_i$ be the valuation of each agent i drawn independently from D_i . We denote by $\mathbf{C} = (C_1, C_2, \dots, C_n)$ the bidding distribution in \mathbf{B} which includes the randomness of both the bidding strategy \mathbf{b} and of the valuations \mathbf{v} . The utility of agent i with valuation v_i can be expressed by $u_i(\mathbf{B}_i(v_i), \mathbf{C}_{-i})$. It should be noted that \mathbf{C}_{-i} does *not* depend on some particular \mathbf{v}_{-i} , but merely on \mathbf{D}_{-i} and \mathbf{B}_{-i} . For any agent i and any subadditive valuation $v_i \in V_i$, consider the deviation $a_i(v_i; \mathbf{w}_{-i}, \mathbf{C}_{-i})$ as defined in Lemma 7, where $\mathbf{w}_{-i} \sim \mathbf{D}_{-i}$. By the definition of the Bayesian Nash equilibrium, we obtain

$$\mathbb{E}_{\mathbf{v}_{-i}} [u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{B}_{-i}(\mathbf{v}_{-i}))] = u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{C}_{-i}) \geq \mathbb{E}_{\mathbf{w}_{-i}} [u_i^{v_i}(a_i(v_i; \mathbf{w}_{-i}, \mathbf{C}_{-i}), \mathbf{C}_{-i})].$$

By taking expectation over v_i and summing up over all agents,

$$\begin{aligned}
& \sum_i \mathbb{E}_{\mathbf{v}} [u_i(\mathbf{B}(\mathbf{v}))] \geq \sum_i \mathbb{E}_{v_i, \mathbf{w}_{-i}} [u_i^{v_i}(a_i(v_i; \mathbf{w}_{-i}, \mathbf{C}_{-i}), \mathbf{C}_{-i})] \\
& = \mathbb{E}_{\mathbf{v}} \left[\sum_i u_i^{v_i}(a_i(\mathbf{v}, \mathbf{C}_{-i}), \mathbf{C}_{-i}) \right] \geq \frac{1}{2} \cdot \sum_i \mathbb{E}_{\mathbf{v}} [v_i(o_i^{\mathbf{v}})] - \sum_i \sum_j \mathbb{E}[b_{ij}]
\end{aligned}$$

So, $\mathbb{E}_{\mathbf{v}}[\text{SW}(\mathbf{B}(\mathbf{v}))] = \sum_i \mathbb{E}_{\mathbf{v}} [u_i(\mathbf{B}(\mathbf{v}))] + \sum_i \sum_j \mathbb{E}[b_{ij}] \geq \frac{1}{2} \cdot \mathbb{E}_{\mathbf{v}}[\text{SW}(\mathbf{o}^{\mathbf{v}})]$. \square

4.2 Simple mechanisms lower bound

Now, we show a lower bound that applies to all simple mechanisms, where the bidding space has size (at most) sub-doubly-exponential in m . More specifically, we apply the general framework of Roughgarden [19], for showing lower bounds on the price of anarchy for *all* simple mechanisms, via communication complexity reductions with respect to the underlying optimization problem. In our setting, the problem is to maximize the social welfare by allocating divisible resources to agents with subadditive valuations. We proceed by proving a communication lower bound for this problem in the following lemma.

Lemma 10. For any constant $\varepsilon > 0$, any $(2 - \varepsilon)$ -approximation (non-deterministic) algorithm for maximizing social welfare in resource allocation problem with subadditive valuations, requires an exponential amount of communication.

Proof. We prove this lemma by reducing the communication lower bound for combinatorial auctions with general valuations (Theorem 3 of [17]) to our setting (see also [7] for a reduction to combinatorial auctions with subadditive agents).

Nisan [17] used an instance with n players and m items, with $n < m^{1/2-\varepsilon}$. Each player i is associated with a set T_i , with $|T_i| = t$ for some $t > 0$. At every instance of this problem, the players' valuations are determined by sets I_i of bundles, where $I_i \subseteq T_i$ for every i . Given I_i , player i 's valuation on some subset S of items is $v_i(S) = 1$, if there exists some $R \in I_i$ such that $R \subseteq S$, otherwise $v_i(S) = 0$. In [17], it was shown that distinguishing between instances with optimal social welfare of n and 1, requires t bits of communication. By choosing t exponential in m , their theorem follows.

We prove the lemma by associating any valuation v of the above combinatorial auction problem, to some appropriate subadditive valuation v' for our setting. For any player i and any fractional allocation $\mathbf{x} = (x_1, \dots, x_m)$, let $A_{x_i} = \{j | x_{ij} > \frac{1}{2}\}$. We define $v'_i(x_i) = v_i(A_{x_i}) + 1$ if $x_i \neq \mathbf{0}$ and $v'_i(x_i) = 0$ otherwise. It is easy to verify that v'_i is subadditive. Notice that $v'_i(x) = 2$ only if there exists $R \in I_i$ such that player i is allocated a fraction higher than $1/2$ for every resource in R . The value $1/2$ is chosen such that no two players are assigned more than that fraction from the same resource. This corresponds to the constraint of an allocation in the combinatorial auction where no item is allocated to two players.

Therefore, in the divisible goods allocation problem, distinguishing between instances where the optimal social welfare is $2n$ and $n + 1$ is equivalent to distinguishing between instances where the optimal social welfare is n and 1 in the corresponding combinatorial auction and hence requires exponential, in m , number of communication bits. \square

The PoA lower bound follows the general reduction described in [19].

Theorem 11. The PoA of ϵ -mixed Nash equilibria⁴ of every simple mechanism, when agents have subadditive valuations, is at least 2.

Remark 12. This result holds only for ϵ -mixed Nash equilibria. Considering exact Nash equilibria, we show a lower bound for all *scale-free* mechanisms in the following section.

4.3 Scale-free mechanisms lower bound

Here we prove a tight lower bound for all scale-free mechanisms including the proportional allocation mechanism. A mechanism (\mathbf{x}, \mathbf{q}) is said to be scale-free if a) for every agent i , resource j and

⁴A bidding profile $\mathbf{B} = \times_i B_i$ is called ϵ -mixed Nash equilibrium if, for every agent i and all bids b'_i , $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(b'_i, \mathbf{b}_{-i})] - \epsilon$.

constant $c > 0$, $x_i(c \cdot \mathbf{b}_j) = x_i(\mathbf{b}_j)$. Moreover, for a fixed \mathbf{b}_{-i} , $x_i(\cdot)$ is non-decreasing and positive whenever b_{ij} is positive. b) The payment for agent i depends only on her bids $b_i = (b_{ij})_j$ and equals to $\sum_{j \in [m]} q_i(b_{ij})$ where $q_i(\cdot)$ is non-decreasing, continuous, normalized ($q_i(0) = 0$), and there always exists a bid b_{ij} such that $q_i(b_{ij}) > 0$.

Theorem 13. The mixed PoA of scale-free mechanisms when agents have subadditive valuations, is at least 2.

Proof. Given a mechanism (\mathbf{x}, \mathbf{q}) , we construct an instance with 2 agents and m resources. Let V be a positive value such that V/m is in the range of both q_1 and q_2 . This can be always done due to our assumptions on q_i . Let T_1 and T_2 be the values such that $q_1(T_1) = q_2(T_2) = V/m$. W.l.o.g. we assume that $T_1 \geq T_2$. By monotonicity of q_1 , $q_1(T_2) \leq V/m$. Pick an arbitrary value $a \in (0, 1)$, and let $h_1 = x_1(a, a)$ and $h_2 = x_2(a, a)$. By the assumption that $x_i(\mathbf{b}_j) > 0$ for $b_{ij} > 0$, we have $h_1, h_2 \in (0, 1)$. Let $v = V/\sqrt{m}$. We define the agents' valuations as:

$$v_1(x) = \begin{cases} 0, & \text{if } \forall j \in [m], x_j = 0, \\ v, & \text{if } \forall j x_j < h_1, \exists k x_k > 0 \\ 2v, & \text{otherwise} \end{cases} \quad v_2(x) = \begin{cases} 0, & \text{if } \forall j \in [m], x_j = 0 \\ V, & \text{if } \exists j x_j < h_2, \exists k x_k > 0 \\ 2V, & \text{otherwise} \end{cases}$$

We claim that the following mixed strategy profile \mathbf{B} is a Nash equilibrium. Agent 1 picks resource l uniformly at random and bids $b_{1l} = y$, and $b_{1k} = 0$, for $k \neq l$, where y is a random variable drawn by the cumulative distribution $G(y) = \frac{mq_2(y)}{V}$, $y \in [0, T_2]$. Agent 2 bids $b_{2j} = z$ for every item j , where z is a random variable drawn from $F(z)$, defined as $F(z) = \frac{v - q_1(T_2) + q_1(z)}{v}$, $z \in [0, T_2]$. Recall that $v = V/\sqrt{m}$ and $q_1(T_2) \leq V/m$. Therefore, $v - q_1(T_2) \geq 0$ and thus $F(0) \geq 0$. Notice that $G(\cdot)$ and $F(\cdot)$ are valid CDFs, due to monotonicity of $q_i(\cdot)$. Since $G(T_2) = 1$, $F(T_2) = 1$ and $q_i(\cdot)$ is continuous, $G(y)$ and $F(y)$ are continuous in $(0, \infty)$ and therefore both functions have no mass point in any $y \neq 0$. We assume that if both agents bid 0 for some resource, agent 2 takes the whole resource. We are ready to show that \mathbf{B} is a Nash equilibrium. For the following arguments notice that $G(T_2) = 1$, $F(T_2) = 1$ and $G(0) = 0$.

If agent 1 bids any y in the range $(0, T_2]$ for a single resource j and zero for the rest, then she gets allocation of at least h_1 (that she values for $2v$), only if $y \geq z$, which happens with probability $F(y)$. This holds due to monotonicity of $x_1(\cdot)$ with respect to y . Otherwise her value is v . Therefore, her expected valuation is $v + F(y)v$. So, for every $y \in (0, T_2]$ her expected utility is $v + F(y)v - q_1(y) = 2v - q_1(T_2)$. If agent 1 picks y according to $G(y)$, her utility is still $2v - q_1(T_2)$, since she bids 0 with zero probability. Suppose agent 1 bids $\mathbf{y} = (y_1, \dots, y_m)$, $y_j \in [0, T_2]$ for every j , with at least two positive bids, and w.l.o.g., assume $y_1 = \max_j y_j$. If $z > y_1$, agent 1 has value v for the allocation she receives. If $z \leq y_1$, agent 1 has value $2v$, but she pays more than $q_1(y_1)$. So, this strategy is dominated by the strategy of bidding y_1 for the first resource and zero for the rest. Bidding greater than T_2 for any resource is dominated by the strategy of bidding exactly T_2 for that resource.

If agent 2 bids $z \in [0, T_2]$ for all resources, she gets an allocation of at least h_2 for all the m resources with probability $G(z)$ (due to monotonicity of $x_2(\cdot)$ with respect to z and to the tie breaking rule). So, her expected utility is $V + G(z)V - mq_2(z) = V$. Bidding greater than T_2 for any resource is dominated by bidding exactly T_2 for this resource. Suppose that agent 2 bids any $\mathbf{z} = (z_1, \dots, z_m)$, with $z_j \in [0, T_2]$ for every j , then, since agent 1 bids positively for any item with probability $1/m$, agent's 2 expected utility is $\frac{1}{m} \sum_j (V + G(z_j)V - \sum_k q_2(z_k)) = \frac{1}{m} \sum_j (V + mq_2(z_j) - \sum_k q_2(z_k)) = \frac{1}{m} (mV + m \sum_j q_2(z_j) - m \sum_k q_2(z_k)) = V$. So, \mathbf{B} is Nash equilibrium.

Therefore, it is sufficient to bound the expected social welfare in \mathbf{B} . Agent 1 bids 0 with zero probability. So, whenever agent 2 bids 0, she receives exactly $m - 1$ resources, which she values for V . Agent 2 bids 0 with probability $F(0) = 1 - \frac{q_1(T_2)}{v} \geq 1 - \frac{V}{mv} = 1 - \frac{1}{\sqrt{m}}$. Hence, $\mathbb{E}[\text{SW}(\mathbf{B})] \leq 2V - F(0) \cdot V + 2v \leq 2V \left(1 + \frac{1}{\sqrt{m}}\right) - V \left(1 - \frac{1}{\sqrt{m}}\right) = V \left(1 + \frac{3}{\sqrt{m}}\right)$. On the other hand, the social welfare in the optimum allocation is $2(V + v) = 2V \left(1 + \frac{1}{\sqrt{m}}\right)$ (agent 1 is allocated h_1 proportion from one resource and the rest is allocated to agent 2). We conclude that $PoA \geq 2 \frac{\left(1 + \frac{1}{\sqrt{m}}\right)}{\left(1 + \frac{3}{\sqrt{m}}\right)}$ which, for large m , converges to 2. \square

5 Budget Constraints

In this section, we switch to scenarios where agents have budget constraints. We use as a benchmark the *effective welfare* similarly to [2, 20]. We compare the effective welfare of the allocation at equilibrium with the optimal effective welfare. We prove an upper bound of $\phi + 1 \approx 2.618$ for coarse correlated equilibria, where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. This improves the previously known 2.78 upper bound in [2] for a single resource and concave valuations.

To prove this upper bound, we use the fact that in the equilibrium there is no profitable unilateral deviation, and, in particular, the utility of agent i obtained by any pure deviating bid a_i should be bounded by her budget c_i , i.e., $\sum_{j \in [m]} a_{ij} \leq c_i$. We define v^c to be the valuation v suppressed by the budget c , i.e., $v^c(x) = \min\{v(x), c\}$. Note that v^c is also subadditive since v is subadditive. For a fixed pair (\mathbf{v}, \mathbf{c}) , let $\mathbf{o} = (o_1, \dots, o_n)$ be the allocation that maximizes the effective welfare. For a fixed agent i and a vector of bids \mathbf{b}_{-i} , we define the vector p_i as $p_i = \sum_{k \neq i} b_k$. We first show the existence of a proper deviation.

Lemma 14. For any subadditive agent i , and any randomized bidding profile \mathbf{B} , there exists a randomized bid $a_i(\mathbf{B}_{-i})$, such that for any $\lambda \geq 1$, it is

$$u_i(a_i(\mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \frac{v_i^{c_i}(o_i)}{\lambda + 1} - \frac{\sum_{j \in [m]} \sum_{k \in [n]} o_{ij} \mathbb{E}[b_{kj}]}{\lambda}.$$

Moreover, for any pure strategy \hat{a}_i in the support of $a_i(\mathbf{B}_{-i})$, $\sum_j \hat{a}_{ij} \leq c_i$.

Proof. In order to find $a_i(\mathbf{B}_{-i})$, we define the truncated bid vector $\tilde{\mathbf{b}}_{-i}$ as follows. For any set $S \subseteq [m]$ of resources, we denote by $\mathbf{1}_S$ the indicator vector w.r.t. S , such that $x_j = 1$ for $j \in S$ and $x_j = 0$ otherwise. For any vector p_i and any $\lambda > 0$, let $T := T(\lambda, p_i)$ be a *maximal* subset of resources such that, $v_i^{c_i}(\mathbf{1}_T) < \frac{1}{\lambda} \sum_{j \in T} o_{ij} p_{ij}$. For every $k \neq i$, if $j \in T$, then $\tilde{b}_{kj} = 0$, otherwise $\tilde{b}_{kj} = b_{kj}$. Similarly, $\tilde{p}_i = \sum_{k \neq i} \tilde{b}_k$. Moreover, if $\mathbf{b}_{-i} \sim \mathbf{B}_{-i}$, then p_i is an induced random variable with distribution denoted by $P_i = \{p_i | \mathbf{b}_{-i} \sim \mathbf{B}_{-i}\}$. We further define distributions $\tilde{\mathbf{B}}_{-i}$ and \tilde{P}_i , as $\tilde{\mathbf{B}}_{-i} = \{\tilde{\mathbf{b}}_{-i} | \mathbf{b}_{-i} \sim \mathbf{B}_{-i}\}$ and $\tilde{P}_i = \{\tilde{p}_i | \tilde{\mathbf{b}}_{-i} \sim \tilde{\mathbf{B}}_{-i}\}$.

Now consider the following bidding strategy $a_i(\mathbf{B}_{-i})$: sampling $b'_i \sim \tilde{P}_i$ and bidding $a_{ij} = \frac{1}{\lambda} o_{ij} b'_{ij}$ for each resource j . We first show $\sum_{j \in [m]} a_{ij} \leq c_i$. It is sufficient to show that $\sum_{j \notin T} a_{ij} \leq v_i^{c_i}(\mathbf{1}_{[m] \setminus T})$ since $v_i^{c_i}(\mathbf{1}_{[m] \setminus T}) \leq c_i$ and $\sum_{j \in T} a_{ij} = 0$. For the sake of contradiction suppose $v_i^{c_i}(\mathbf{1}_{[m] \setminus T}) < \sum_{j \notin T} a_{ij}$. Then, by the definition of T and \tilde{p}_i , $v_i^{c_i}(\mathbf{1}_{[m]}) \leq v_i^{c_i}(\mathbf{1}_T) + v_i^{c_i}(\mathbf{1}_{[m] \setminus T}) < \frac{1}{\lambda} \sum_{j \in T} o_{ij} p_{ij} + \sum_{j \notin T} a_{ij} = \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} p_{ij}$, which contradicts the maximality of T .

Next we show for any bid b_i and $\lambda > 0$,

$$v_i^{c_i}(x_i(b_i, \mathbf{B}_{-i})) + \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \mathbb{E}_{p_i \sim P_i} [p_{ij}] \geq v_i^{c_i}(x_i(b_i, \tilde{\mathbf{B}}_{-i})) + \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \mathbb{E}_{\tilde{p}_i \sim \tilde{P}_i} [\tilde{p}_{ij}] \quad (1)$$

Observe that $x_i(b_i, \tilde{\mathbf{b}}_{-i}) \leq x_i(b_i, \mathbf{b}_{-i}) + \mathbf{1}_T$. Therefore, and by the definitions of T and \tilde{p}_i ,

$$\begin{aligned} v_i^{c_i}(x_i(b_i, \tilde{\mathbf{b}}_{-i})) &\leq v_i^{c_i}(x_i(b_i, \mathbf{b}_{-i})) + v_i^{c_i}(\mathbf{1}_T) \leq v_i^{c_i}(x_i(b_i, \mathbf{b}_{-i})) + \frac{1}{\lambda} \sum_{j \in T} o_{ij} p_{ij} \\ &= v_i^{c_i}(x_i(b_i, \mathbf{b}_{-i})) + \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} p_{ij} - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \tilde{p}_{ij}. \end{aligned}$$

The claim follows by rearranging terms and taking the expectation of \mathbf{b}_{-i} , $\tilde{\mathbf{b}}_{-i}$, p_i and \tilde{p}_i over \mathbf{B}_{-i} , $\tilde{\mathbf{B}}_{-i}$, P_i and \tilde{P}_i , respectively. We are now ready to prove the statement of the lemma.

$$\begin{aligned} \mathbb{E}_{b'_i \sim \tilde{P}_i} \left[u_i \left(\frac{1}{\lambda} o_i b'_i, \mathbf{B}_{-i} \right) \right] &= \mathbb{E}_{b'_i \sim \tilde{P}_i} \left[v_i \left(\frac{1}{\lambda} o_i b'_i, \mathbf{B}_{-i} \right) \right] - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \mathbb{E}_{b'_i \sim \tilde{P}_i} [b'_{ij}] \\ &\geq \mathbb{E}_{b'_i \sim \tilde{P}_i} \left[v_i^{c_i} \left(\frac{1}{\lambda} o_i b'_i, \mathbf{B}_{-i} \right) \right] - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \mathbb{E}_{\tilde{p}_i \sim \tilde{P}_i} [\tilde{p}_{ij}] \quad (\text{by definition of } v_i^{c_i}) \\ &\geq \mathbb{E}_{b'_i \sim \tilde{P}_i} \left[v_i^{c_i} \left(\frac{1}{\lambda} o_i b'_i, \tilde{\mathbf{B}}_{-i} \right) \right] - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \mathbb{E}_{p_i \sim P_i} [p_{ij}] \quad (\text{by Inequality (1)}) \\ &\geq \frac{1}{2} \mathbb{E}_{b'_i \sim \tilde{P}_i} \mathbb{E}_{\tilde{p}_i \sim \tilde{P}_i} \left[v_i^{c_i} \left(\frac{o_i b'_i}{o_i b'_i + \lambda \tilde{p}_i} + \frac{o_i \tilde{p}_i}{o_i \tilde{p}_i + \lambda b'_i} \right) \right] - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \sum_{k \neq i} B_{kj} \\ &\quad (\text{by swapping } b'_i \text{ with } \tilde{p}_i \text{ and the subadditivity of } v_i^{c_i}(\cdot)) \\ &\geq \frac{1}{2} \mathbb{E}_{b'_i \sim \tilde{P}_i} \mathbb{E}_{\tilde{p}_i \sim \tilde{P}_i} \left[v_i^{c_i} \left(o_i \left(\frac{b'_i}{b'_i + \lambda \tilde{p}_i} + \frac{\tilde{p}_i}{\tilde{p}_i + \lambda b'_i} \right) \right) \right] - \frac{1}{\lambda} \sum_{j \in [m]} \sum_{k \in [n]} o_{ij} \mathbb{E}[b_{kj}] \\ &\geq \frac{1}{2} v_i^{c_i} \left(\frac{2o_i}{\lambda + 1} \right) - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \sum_k B_{kj} \quad (\text{by monotonicity of } v_i^{c_i}) \\ &\geq \frac{1}{\lambda + 1} v_i^{c_i}(o_i) - \frac{1}{\lambda} \sum_{j \in [m]} o_{ij} \sum_k B_{kj} \quad (\text{subadditivity of } v_i^{c_i}; \frac{2}{\lambda + 1} \leq 1) \end{aligned}$$

For the second inequality, notice that the second term doesn't depend on b'_i , so we apply Lemma 11 for every b'_i . For the forth and fifth inequalities, $o_i \leq 1$ and $\frac{b'_i}{b'_i + \lambda \tilde{p}_i} + \frac{\tilde{p}_i}{\tilde{p}_i + \lambda b'_i} \geq \frac{2}{\lambda + 1}$ for every b'_i , \tilde{p}_i and $\lambda \geq 1$. \square

We are ready to show the PoA bound by using the above lemma.

Theorem 15. The coarse correlated PoA for the proportional allocation mechanism when agents have budget constraints and subadditive valuations, is at most $\phi + 1 \approx 2.618$.

Proof. Suppose \mathbf{B} is a coarse correlated equilibrium. Let A be the set of agents such that for every $i \in A$, $v_i(\mathbf{B}) \leq c_i$. For simplicity, we use $v_i^{c_i}(\mathbf{B})$ to denote $\min\{\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[v_i(x_i(\mathbf{b}))], c_i\}$. Then for all $i \notin A$, $v_i^{c_i}(\mathbf{B}) = c_i \geq v_i^{c_i}(o_i)$ and $v_i^{c_i}(\mathbf{B}) = c_i \geq \sum_{j \in [m]} \mathbb{E}[b_{ij}]$. The latter inequality comes from that agents do not bid higher than their budgets. Let $\lambda = \phi$. So $1 - 1/\lambda = 1/(1 + \lambda)$. By taking the linear combination and summing up over all agents not in A , we get

$$\sum_{i \notin A} v_i^{c_i}(\mathbf{B}) \geq \frac{1}{\lambda + 1} \sum_{i \notin A} v_i^{c_i}(o_i) + \frac{1}{\lambda} \sum_{i \notin A} \sum_{j \in [m]} \mathbb{E}[b_{ij}] \quad (2)$$

For every $i \in A$, we consider the deviating bidding strategy $a_i(\mathbf{B}_{-i})$ that is described in Lemma 14, then

$$\begin{aligned} v_i^{c_i}(\mathbf{B}) &= v_i(x_i(\mathbf{B})) = u_i(x_i(\mathbf{B})) + \sum_{j \in [m]} \mathbb{E}[b_{ij}] \geq u_i(a_i(\mathbf{B}_{-i}), \mathbf{B}_{-i}) + \frac{1}{\lambda} \sum_{j \in [m]} \mathbb{E}[b_{ij}] \\ &\geq \frac{1}{\lambda+1} v_i^{c_i}(o_i) - \frac{1}{\lambda} \sum_{j \in [m]} \sum_{k \in [n]} o_{ij} \mathbb{E}[b_{kj}] + \frac{1}{\lambda} \sum_{j \in [m]} \mathbb{E}[b_{ij}] \end{aligned}$$

By summing up over all $i \in A$ and by combining with inequality (2) we get

$$\begin{aligned} &\sum_{i \in [n]} \min\{v_i(x_i(\mathbf{B})), c_i\} \\ &\geq \frac{1}{\lambda+1} \sum_{i \in [n]} v_i^{c_i}(o_i) + \frac{1}{\lambda} \sum_{i \in [n]} \sum_{j \in [m]} \mathbb{E}[b_{ij}] - \frac{1}{\lambda} \sum_{i \in A} \sum_{j \in [m]} \sum_{k \in [n]} o_{ij} \mathbb{E}[b_{kj}] \\ &\geq \frac{1}{\lambda+1} \sum_{i \in [n]} v_i^{c_i}(o_i) \quad \left(\text{since } \sum_{i \in A} o_{ij} \leq 1 \right) \end{aligned}$$

Therefore, the PoA with respect to the effective welfare is at most $\phi + 1$. (recall that for Inequality (2) we set $\lambda = \phi$) \square

By applying Jensen's inequality for concave functions, our upper bound also holds for the Bayesian case with single-resource and concave functions.

Theorem 16. The Bayesian PoA of single-resource proportional allocation games is at most $\phi+1 \approx 2.618$, when agents have budget constraints and concave valuations.

Proof. Suppose \mathbf{B} is a Bayesian Nash equilibrium. Recall that in the Bayesian setting, agent i 's type $t_i = (v_i, c_i)$ are drawn from some know distribution independently. We use the notation $\mathbf{C} = (C_1, C_2, \dots, C_n)$ to denote the bidding distribution in \mathbf{B} which includes the randomness of bidding strategy \mathbf{b} and agents' types \mathbf{t} , that is $b_i(t_i) \sim C_i$. Then the utility of agent i with type t_i is $u_i(B_i(t_i), \mathbf{C}_{-i})$. Notice that \mathbf{C}_{-i} does not depend on any particular \mathbf{t}_{-i} .

Recall that $v^c(x) = \min\{v(x), c\}$. It is easy to check v^c is concave if v is concave. For any agents types $\mathbf{t} = (\mathbf{v}, \mathbf{c})$, let $\mathbf{o}^{\mathbf{t}} = (o_1^{\mathbf{t}}, \dots, o_n^{\mathbf{t}})$ be the allocation vector that maximizes the *effective welfare*. We define $o_i^{t_i}$ to be the expected allocation over $\mathbf{t}_{-i} \sim \mathbf{D}_{-i}$ to agent i , in the optimum solution with respect to effective welfare, when her type is t_i . Formally, $o_i^{t_i} = \mathbb{E}_{\mathbf{t}_{-i} \sim \mathbf{D}_{-i}}[o_i^{(t_i, \mathbf{t}_{-i})}]$.

For all agent i , let A_i be the set of t_i such that $v_i(x_i(B_i(t_i), \mathbf{C}_{-i})) \leq c_i$. For simplicity, we use to $v_i^{c_i}(B_i(t_i), \mathbf{C}_{-i})$ to denote $\min\{\mathbb{E}_{\mathbf{t}_{-i}, \mathbf{b} \sim \mathbf{B}(\mathbf{t})}[v_i(x_i(\mathbf{b}))], c_i\}$. For every $t_i \notin A_i$, $v_i^{c_i}(B_i(t_i), \mathbf{C}_{-i}) = c_i \geq \min\{\mathbb{E}_{\mathbf{t}_{-i}}[v_i(o_i^{\mathbf{t}})], c_i\}$ and $v_i^{c_i}(B_i(t_i), \mathbf{C}_{-i}) = c_i \geq \mathbb{E}[b_i(t_i)]$. The latter inequality comes from that agents do not bid above their budget. Let $\lambda = \phi$. So $1 - 1/\lambda = 1/(1 + \lambda)$. By taking the linear combination, taking the expectation over all $t_i \notin A_i$ and summing up over all agents not in A , we get

$$\sum_i \mathbb{E}_{t_i \notin A_i} [v_i^{c_i}(B_i(t_i), \mathbf{C}_{-i})] \geq \sum_i \mathbb{E}_{t_i \notin A_i} \left[\frac{1}{\lambda+1} \min \left\{ \mathbb{E}_{\mathbf{t}_{-i}} [v_i(o_i^{\mathbf{t}})], c_i \right\} + \frac{1}{\lambda} b_i(t_i) \right] \quad (3)$$

For every $t_i \in A_i$, by Lemma 14, there exists a randomized bid $a_i(t_i, \mathbf{B}_{-i})$ for agent i , such that,

for any $\lambda \geq 1$: $u_i(a_i(t_i, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \frac{1}{\lambda+1} v_i^{c_i}(o_i^{t_i}) - \frac{1}{\lambda} o_i^{t_i} \sum_{k \neq i} \mathbb{E}[b_k]$. By the definition of equilibria,

$$\begin{aligned} & v_i^{c_i}(B_i(t_i), \mathbf{C}_{-i}) = v_i(B_i(t_i), \mathbf{C}_{-i}) = u_i(B_i(t_i), \mathbf{C}_{-i}) + \mathbb{E}[b_i(t_i)] \\ & \geq u_i(a_i(t_i, \mathbf{C}_{-i}), \mathbf{C}_{-i}) + \frac{1}{\lambda} B_i(t_i) \geq \frac{1}{\lambda+1} v_i^{c_i}(o_i^{t_i}) - \frac{1}{\lambda} o_i^{t_i} \sum_{k \in [n]} \mathbb{E}[b_k] + \frac{1}{\lambda} B_i(t_i) \\ & \geq \frac{1}{\lambda+1} \min \left\{ \mathbb{E}_{\mathbf{t}_{-i}} [v_i(o_i^{\mathbf{t}})], c_i \right\} - \frac{1}{\lambda} o_i^{t_i} \sum_{k \in [n]} \mathbb{E}[b_k] + \frac{1}{\lambda} \mathbb{E}[b_i(t_i)] \end{aligned}$$

The last inequality holds due to Jensen's inequality for concave functions. By taking the expectation over all $t_i \in A_i$, summing over all agents and combining with inequality (3):

$$\begin{aligned} & \sum_i \mathbb{E}_{t_i} [v_i^{c_i}(B_i(t_i), \mathbf{C}_{-i})] \\ & \geq \frac{1}{\lambda+1} \sum_i \mathbb{E}_{t_i} \left[\min \left\{ \mathbb{E}_{\mathbf{t}_{-i}} [v_i(o_i^{\mathbf{t}})], c_i \right\} \right] + \frac{1}{\lambda} \sum_i \mathbb{E}[b_i] - \frac{1}{\lambda} \sum_i \mathbb{E}_{t_i \in A_i} [o_i^{t_i}] \sum_{k \in [n]} \mathbb{E}[b_k] \\ & \geq \frac{1}{\lambda+1} \sum_i \mathbb{E}_{t_i} \left[\min \left\{ \mathbb{E}_{\mathbf{t}_{-i}} [v_i(o_i^{\mathbf{t}})], c_i \right\} \right] + \frac{1}{\lambda} \sum_i \mathbb{E}[b_i] - \frac{1}{\lambda} \sum_{k \in [n]} \mathbb{E}[b_k] \\ & = \frac{1}{\lambda+1} \sum_i \mathbb{E}_{t_i} \left[\min \left\{ \mathbb{E}_{\mathbf{t}_{-i}} [v_i(o_i^{\mathbf{t}})], c_i \right\} \right] \end{aligned}$$

The first inequality is due to that $\sum_i \mathbb{E}_{t_i} [o_i^{t_i}] = \sum_i \mathbb{E}_{\mathbf{t}} [o_i^{\mathbf{t}}] = \mathbb{E}_{\mathbf{t}} [\sum_i o_i^{\mathbf{t}}] \leq 1$, since for every \mathbf{t} , $\sum_i o_i^{\mathbf{t}} \leq 1$. Therefore, the PoA is at most $\phi + 1$. \square

Remark 17. Syrgkanis and Tardos [20], compared the social welfare in the equilibrium with the effective welfare in the optimum allocation. Caragiannis and Voudouris [2] also give an upper bound of 2 for this ratio in the single resource case. We can obtain the same upper bound by replacing λ with 1 in Lemma 14 and following the ideas of Theorems 8 and 9.

6 Polyhedral Environment

In this section, we study the efficiency of the proportional allocation mechanism in the polyhedral environment, that was previously studied by Nguyen and Tardos [16]. We show a *tight* price of anarchy bound of 2 for agents with subadditive valuations. Recall that, in this setting, the allocation to each agent i is now represented by a *single parameter* x_i , and not by a vector (x_{i1}, \dots, x_{im}) . In addition, any feasible allocation vector $\mathbf{x} = (x_1, \dots, x_n)$ should satisfy a polyhedral constraint $A \cdot \mathbf{x} \leq \mathbf{1}$, where A is a non-negative $m \times n$ matrix and each row of A corresponds to a different resource, and $\mathbf{1}$ is a vector with all ones. Each agent aims to maximize her utility $u_i = v_i(x_i) - q_i$, where v_i is a subadditive function representing the agent's valuation. The proportional allocation mechanism determines the following allocation and payments for each agent:

$$x_i(\mathbf{b}) = \min_{j: a_{ij} > 0} \left\{ \frac{b_{ij}}{a_{ij} \sum_{k \in [n]} b_{kj}} \right\}; \quad q_i(\mathbf{b}) = \sum_{j \in [m]} b_{ij},$$

where a_{ij} is the (i, j) -th entry of matrix A . It is easy to verify that the above allocation satisfies the polyhedral constraints.

Theorem 18. If agents have subadditive valuations, the pure PoA of the proportional allocation mechanism in the polyhedral environment is exactly 2.

Proof. We first show that the PoA is at most 2. Let $\mathbf{o} = \{o_1, \dots, o_n\}$ be the optimal allocation, \mathbf{b} be a pure Nash Equilibrium, and let $p_{ij} = \sum_{k \neq i} b_{kj}$. For each agent i , consider the deviating bid b'_i such that $b'_{ij} = o_i a_{ij} p_{ij}$ for all resources j . Since \mathbf{b} is a Nash Equilibrium,

$$\begin{aligned} u_i(\mathbf{b}) &\geq u_i(b'_i, b_{-i}) = v_i \left(\min_{j: a_{ij} > 0} \left\{ \frac{o_i a_{ij} p_{ij}}{a_{ij} (p_{ij} + o_i a_{ij} p_{ij})} \right\} \right) - \sum_{j \in [m]} o_i a_{ij} p_{ij} \\ &\geq v_i \left(\frac{o_i}{2} \right) - \sum_{j \in [m]} o_i a_{ij} p_{ij} \geq \frac{1}{2} v_i(o_i) - \sum_{j \in [m]} o_i a_{ij} p_{ij} \end{aligned}$$

The second inequality is true since $A \cdot \mathbf{x} \leq \mathbf{1}$, for every allocation \mathbf{x} , and therefore $o_i a_{ij} < 1$. The last inequality holds due to subadditivity of v_i . By summing up over all agents, we get

$$\sum_i u_i(\mathbf{b}) \geq \frac{1}{2} \sum_i v_i(o_i) - \sum_{j \in [m]} \sum_{i \in [n]} o_i a_{ij} p_{ij} \geq \frac{1}{2} \sum_i v_i(o_i) - \sum_{j \in [m]} \sum_{k \in [n]} b_{kj}.$$

The last inequality holds due to the fact that $p_{ij} \leq \sum_{k \in [n]} b_{kj}$ and $\sum_{i \in [n]} o_i a_{ij} \leq 1$. The fact that $\text{PoA} \leq 2$ follows by rearranging the terms.

For the lower bound, consider a game with only two agents and a single resource where the polyhedral constraint is given by $x_1 + x_2 \leq 1$. The valuation of the first agent is $v_1(x) = 1 + \epsilon \cdot x$, for some $\epsilon < 1$ if $x < 1$ and $v_1(x) = 2$ if $x = 1$. The valuation of the second agent is $\epsilon \cdot x$. One can verify that these two functions are subadditive and the optimal social welfare is 2. Consider the bidding strategies $b_1 = b_2 = \frac{\epsilon}{4}$. The utility of agent 1, when she bids x and agent 2 bids $\frac{\epsilon}{4}$, is given by $1 + \epsilon \cdot \frac{x}{x + \epsilon/4} - x$ which is maximized for $x = \frac{\epsilon}{4}$. The utility of agent 2, when she bids x and agent 1 bids $\frac{\epsilon}{4}$, is $\epsilon \cdot \frac{x}{x + \epsilon/4} - x$ which is also maximized when $x = \frac{\epsilon}{4}$. So (b_1, b_2) is a pure Nash Equilibrium with social welfare $1 + \epsilon$. Therefore, the PoA converges to 2 when ϵ goes to 0. \square

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